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Paraconformal geometry of *n*th-order ODEs, and exotic holonomy in dimension four

Maciej Dunajski [a,*], Paul Tod [b]

[a] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*
[b] *The Mathematical Institute, Oxford University, 24-29 St. Giles, Oxford OX1 3LB, UK*

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Abstract

We characterise *n*th-order ODEs for which the space of solutions M is equipped with a particular paraconformal structure in the sense of Bailey and Eastwood [T.N. Bailey, M.G. Eastwood, Complex Paraconformal manifolds, their differential geometry and twistor theory, Forum Math. 3 (1991) 61–103], that is a splitting of the tangent bundle as a symmetric tensor product of rank-two vector bundles. This leads to the vanishing of $(n-2)$ quantities constructed from of the ODE.

If $n=4$ the paraconformal structure is shown to be equivalent to the exotic $\mathcal{G}_3$ holonomy of Bryant. If $n=4$, or $n\geq 6$ and M admits a torsion-free connection compatible with the paraconformal structure then the ODE is trivialisable by point or contact transformations, respectively.

If $n=2$ or $3M$ admits an affine paraconformal connection with no torsion. In these cases additional constraints can be imposed on the ODE so that M admits a projective structure if $n=2$, or an Einstein–Weyl structure if $n=3$. The third-order ODE can in this case be reconstructed from the Einstein–Weyl data.

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1. Introduction

Consider a relation of the form:

$$\Psi(x, y, t) = 0 \tag{1.1}$$

* Corresponding author.
E-mail addresses: M.Dunajski@damtp.cam.ac.uk (M. Dunajski); tod@maths.ox.ac.uk (P. Tod).

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between the variables $t = (t^1, t^2, \ldots, t^n)$ (local coordinates on an n-dimensional manifold M), and (x, y) (local coordinates on a two-dimensional manifold $\mathcal{Z}$, which we shall call the twistor space). For each fixed choice of (x, y)m relation (1.1) defines a hypersurface in M. Conversely each choice of t defines a curve L_t in $\mathcal{Z}$. Given conditions on the derivatives of Ψ, we can apply the implicit function theorem to (1.1), and regard L_t as a graph:

$$x \longrightarrow (x, y = Z(x, t)). \tag{1.2}$$

Consider the system of algebraic equations consisting of $y = Z(x, t)$, and the first $(n - 1)$ derivatives with respect to x. Solving this system for t, and differentiating once more with respect to x yields:

$$y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F(x, y, y', \ldots, y^{(n-1)}), \tag{1.3}$$

where the explicit form of F is completely determined by (1.1). This procedure will lead to an nth (as opposed to a lower order) order ODE if Ψ is sufficiently smooth and non-degenerate in an suitable sense. This open non-degeneracy condition is best expressed in terms of $Z(x, t)$ by demanding that the gradients $\nabla Z, \nabla Z', \ldots, \nabla Z^{(n-1)}$ with respect to t are linearly independent on M.

To achieve a more geometric picture we define the $(n + 1)$-dimensional hyper-surface $\mathcal{F} \subset \mathcal{Z} \times M$ called the correspondence space by the following incidence relation:

$$\mathcal{F} = \{((x, y), t) \in \mathcal{Z} \times M | z \in L_t\}. \tag{1.4}$$

The double fibration

$$M \xleftarrow{p} \mathcal{F} \xrightarrow{q} \mathcal{Z} \tag{1.5}$$

is then defined by relation (1.1), and therefore by ODE (1.3).

Putting various geometric structures on M (which from now on will be identified with the space of solutions to ODE (1.3)) imposes additional constraints on F. This idea goes back to Cartan [6], and his program of 'geometrising' ODEs. Extending Cartan's program to PDEs is possible, and underlies some approaches to general relativity [13], and other problems in mathematical physics [10].

A different approach based on twistor theory was suggested by Hitchin [14]. In this approach one works in the holomorphic category and (x, y, t) are complex numbers. The graph (1.2) represents a compact holomorphic (i.e. rational) curve in $\mathcal{Z}$ with a prescribed normal bundle. The local differential geometry of M is encoded in global embeddings of the family of curves (parameterized by t) in $\mathcal{Z}$. In this approach, ODE (1.3) does not explicitly appear in the correspondence between M and $\mathcal{Z}$. The details of Hitchin's construction and its connection with the ODE approach have partially been worked out only for $n = 2$ [16]. In this case there exists an embedding of a rational curve with normal bundle $\mathcal{O}(1)$ in $\mathcal{Z}$ if and only if

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} F_{11} - 4\frac{\mathrm{d}}{\mathrm{d}x} F_{01} - F_1 \frac{\mathrm{d}}{\mathrm{d}x} F_{11} + 4F_1 F_{01} - 3F_0 F_{11} + 6F_{00} = 0, \tag{1.6}$$

where

$$F_0 = \frac{\partial F}{\partial y}, \quad F_1 = \frac{\partial F}{\partial y'}, \quad F_2 = \frac{\partial F}{\partial y''}, \ldots, F_{n-1} = \frac{\partial F}{\partial y^{(n-1)}},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \sum_{k=1}^{n-1} y^{(k)} \frac{\partial}{\partial y^{(k-1)}} + F \frac{\partial}{\partial y^{(n-1)}}.$$

The two-dimensional moduli space M of $\mathcal{O}(1)$ curves is in this case equipped with a projective structure, in the sense that the hyper-surfaces (curves) of constant (x, y) in (1.1) are geodesics of a torsion-free connection. Conversely, given a projective structure on M one defines $\mathcal{Z}$ as the quotient space of the foliation of $\mathbb{P}(TM)$ by the orbits of the geodesic flow. Each projective tangent space $\mathbb{P}(T_t M)$ maps to a rational curve with self-intersection number one in $\mathcal{Z}$.

The case $n = 3$ goes back to Cartan [6] and Chern [8], and was recently revisited in [18]. The conformal structure on M is defined by demanding that hyper-surfaces $\Sigma \subset M$ corresponding to points in $\mathcal{Z}$ are null. This conformal structure is well defined if $F(x, y, y', y'')$ satisfies a third-order differential constraint:

$$\frac{1}{3} F_2 \frac{\mathrm{d}}{\mathrm{d}x} F_2 - \frac{1}{6} \frac{\mathrm{d}^2}{\mathrm{d}x^2} F_2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} F_1 - \frac{2}{27} (F_2)^3 - \frac{1}{3} F_2 F_1 - F_0 = 0. \tag{1.7}$$

This constraint has already appeared in the work of Wünschmann [20]. The hyper-surfaces Σ are totally geodesic with respect to some torsion-free connection D if F satisfies the additional condition:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} F_{22} - \frac{\mathrm{d}}{\mathrm{d}x} F_{12} + F_{02} = 0. \tag{1.8}$$

The existence of a two-parameter family of totally geodesic null hypersurface in M is equivalent to the vanishing of the trace-free part of the symmetrised Ricci tensor of D. This is the Einstein–Weyl condition first introduced in [7]. The three-dimensional Einstein–Weyl spaces can therefore be obtained from a particular class of third-order ODEs (1.3). In the twistor approach [14] the moduli space of rational curves in $\mathcal{Z}$ with normal bundle $\mathcal{O}(2)$ is automatically equipped with an EW structure, and all analytic EW structures locally arise in such a way.

The only other case which has attracted some attention is $n = 4$. Bryant [3] has shown that there exists a correspondence between a class of fourth-order ODEs, and exotic non-metric holonomies in dimension four. The conditions on F are only implicit in Bryant's work.

In Section 2 we shall generalise the Wünschmann condition (1.7) to $(n - 2)$ conditions in the case of nth-order ODEs, and give an example of an ODE for which all these conditions are satisfied.

Definition 1.1. A paraconformal structure on a smooth manifold M is a bundle isomorphism:

$$TM \cong \mathbb{S} \odot \mathbb{S} \odot \cdots \odot \mathbb{S} = S^{n-1}(\mathbb{S}), \tag{1.9}$$

where $\mathbb{S} \to M$ is a real rank-two vector bundle, and $\odot$ denotes symmetric tensor product.

More general paraconformal structures have been considered in [2,1] (where they were called almost Grassmann structures) but we shall only work with (1.9). The isomorphism (1.9) identifies each tangent space $T_t M$ with the space of homogeneous $(n - 1)$th-order polynomials in two variables. The vectors corresponding to polynomials with repeated root of multiplicity $(n - 1)$ are called maximally null. A hypersurface in M is maximally null if its normal vector is maximally null. All results established in this paper are local on M.

In the next section, we shall prove the following theorem.

Theorem 1.2. *Assume that the space of solutions M to the nth-order ODE (1.3) is equipped with a paraconformal structure (1.9) such that the two-parameter family of hypersurfaces (1.1) are maximally null. Then F satisfies* $(n-2)$ *conditions of the form*:

$$C_k\left(F_i, \frac{\mathrm{d}F_i}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}F_i}{\mathrm{d}x^{n-1}}\right) = 0, \quad i = 0, \ldots, n-1, \; k = 1, \ldots, n-2. \tag{1.10}$$

Each expression C_k is a polynomial in the derivatives of F of order less than or equal to n, and Eq. (1.10) *are invariant under point transformations on $\mathcal{Z}$ (i.e. transformations induced by a change of variables $\hat{x} = \hat{x}(x, y)$, $\hat{y} = \hat{y}(x, y)$).*

Conversely given (1.10) there is a paraconformal structure on M such that points in $\mathcal{Z}$ define maximally null hypersurfaces in M.

In Section 3 we shall relate the paraconformal structure on M to the existence of a subbundle of $\mathbb{P}(TM)$ with rational normal curves as fibres, and demonstrate that conditions (1.10) are the classical Wilczynski invariants [19] on the linearisation of (1.3) at a fixed solution. This strengthens Theorem (1.2) as the Wilczynski conditions are invariant under the wider class of contact transformation.

In Section 4 we shall show that the paraconformal structure (1.9) exists on the moduli space of rational curves with normal bundle $\mathcal{O}(n-1)$ in a complex surface, which leads to a twistorial interpretation of the constraints on F.

In Section 5 we shall discuss the case $n = 4$, where the existence of the paraconformal structure is equivalent to the existence of the torsion-free connection with $\mathcal{G}_3$ holonomy on the space of solutions to (1.3).

Theorem 1.3. *Let M be the space of solutions to the fourth-order ODE*:

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = F(x, y, y', y'', y''').$$

The following conditions are equivalent:

1. *M admits the paraconformal structure (1.9) with maximally null surfaces (1.1).*
2. *M admits a torsion-free connection with holonomy $\mathcal{G}_3$.*
3. *F satisfies a pair of third-order PDEs*:

$$\frac{11}{1600}(F_3)^4 - \frac{9}{50}(F_3)^2\frac{\mathrm{d}}{\mathrm{d}x}F_3 - \frac{1}{200}(F_3)^2F_2 + \frac{21}{100}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)^2 + \frac{1}{50}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)F_2$$
$$- \frac{9}{100}(F_2)^2 + \frac{7}{20}F_3\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3 - \frac{1}{5}\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_3 + \frac{3}{10}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_2 - \frac{1}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_2 - F_0 = 0, \tag{1.11}$$

$$\frac{9}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_3 - \frac{3}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3 + 3\frac{\mathrm{d}}{\mathrm{d}x}F_2 - \frac{3}{8}(F_3)^3 - \frac{3}{2}F_2F_3 - 3F_1 = 0. \tag{1.12}$$

(*differentiating the second condition w.r.t. x and subtracting its constant multiple from the first one leads to a couple of third-order PDEs for F*).

In Section 6 we shall study connections preserving the paraconformal structure, and show that they must necessarily have torsion if $n = 4$ or $n \geq 6$ (we stress that our definition of the torsion-free paraconformal connection is stronger that Bryant's torsion-free $\mathcal{G}_3$ holonomy [3]).

Theorem 1.4. *If M admits the paraconformal structure (1.9) and*

$$D : \Gamma(\mathbb{S}^k) \longrightarrow \Gamma(\mathbb{S}^k \otimes T^*M) = \Gamma(\mathbb{S}^{k+n-1}),$$

where $\mathbb{S}^k = \mathbb{S}^{\otimes k}$, $k = 0, 1, \ldots, n-1$, *is a torsion-free connection preserving the paraconformal structure then additional constraints (6.5), (A.21)–(A.25), with* (P, Q) *given by (2.6), need to be satisfied.*

In particular, ODE (1.3) can be transformed to

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0$$

by point transformations if $n = 4$, *and by contact transformations if* $n \geq 6$. *If* $n < 4$ *then D always exists.*

The case $n = 5$ is special, and we shall give an example (Eq. (6.9)) of a fifth-order ODE such that M admits a paraconformal structure with a paraconformal torsion-free connection.

If $n = 3$ the paraconformal structure (1.9) is conformal in the usual sense. The rank-two vector bundle $\mathbb{S}$ in isomorphism (1.9) is in this case the usual spin bundle and its sections are called the two-component spinors. We shall continue to call it the spin bundle in the general case $n > 3$ although its sections are not spinors as there is no underlying orthogonal group.

In Section 7 we shall concentrate on the case $n = 3$, where condition (1.10) for F is the Wünschmann condition (1.7). We shall give an algorithm for determining the third-order ODE satisfying (1.8), and (1.7) from a given Einstein–Weyl structure, based on the Lax formulation of the Einstein–Weyl conditions.

Most calculations leading to invariants like (1.11) and (1.12) were performed (or checked) using MAPLE. The resulting long expressions are usually unilluminating. They are nevertheless useful in constructing explicit examples like (2.7), and we have decided to include them in the paper. Readers who want to verify our calculations can obtain the MAPLE programs from us.

2. Wünschmann invariants

In this Section, we shall establish Theorem 1.2, and give an example of an ODE which leads to a paraconformal structure for any n. First we need to introduce some notation. The isomorphism (1.9) identifies each tangent space $T_t M$ with the space of homogeneous $(n-1)$th-order polynomials in two variables

$$T \in TM \longrightarrow \mathbf{t} = T^{A_1 A_2 \cdots A_{n-1}} z_{A_1} z_{A_2} \cdots z_{A_{n-1}}, \qquad A_1, A_2, \ldots, A_{n-1} = 0, 1,$$

where $z_{A_i} = (z_0, z_1) \in \mathbb{R}^2$, and $T^{A_1 A_2 \cdots A_{n-1}}$ is symmetric in its indices.

Our considerations are local on M so we choose a trivialisation of TM and represent a vector T by its components T^a, $a = 1, \ldots, n$ with respect to some basis. We also choose a trivalisation of $\mathbb{S}$. The paraconformal structure is then defined in terms of van der Waerden symbols $\sigma^a_{A_1 \cdots A_{n-1}}$ (which are symmetric in $A_1, \ldots, A_{n-1}$) by

$$T^a = \sigma^a_{A_1 \cdots A_{n-1}} T^{A_1 A_2 \cdots A_{n-1}}.$$

The bold letters denote homogeneous polynomials. The summation convention is used unless stated otherwise. For any $0 \leq r \leq n-1$, we define $V_r \subset \mathbb{R}[z_0, z_1]$ to be the $(r+1)$-dimensional space of homogeneous polynomials of degree r. Let $\mathbf{t} \in V_{n-1}$. The space V_{n-1} is an $SL(2, \mathbb{R})$ module, and the infinitesimal action of $SL(2, \mathbb{R})$ is generated by $\mathbf{t} \to H(\mathbf{t})$, where

$$H = H_A{}^B z_B \frac{\partial}{\partial z_A} \in \mathbf{sl}(2, \mathbb{R}),$$

and $H_A{}^B$ is one of the following matrices:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For each $p \geq 0$ define a linear, $SL(2, \mathbb{R})$ equivariant mapping $V_r \otimes V_s \longrightarrow V_{r+s-2p}$ given by

$$\langle \mathbf{t}, \mathbf{s} \rangle_p = \varepsilon_{A_1 B_1} \varepsilon_{A_2 B_2} \cdots \varepsilon_{A_p B_p} \frac{\partial^p \mathbf{t}}{\partial z_{A_1} \cdots \partial z_{A_p}} \frac{\partial^p \mathbf{s}}{\partial z_{B_1} \cdots \partial z_{B_p}}, \tag{2.1}$$

where

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a symplectic form on the real fibres of $\mathbb{S}$. In the following sections, we will not fix this symplectic form. The anti-symmetric matrix ε will only be defined up to scale, and each choice of the scale will provide an identification between $\mathbb{S}$ and its dual bundle.

In particular $\langle , \rangle_{n-1} : V_{n-1} \times V_{n-1} \to \mathbb{R}$ is a symmetric or skew-symmetric (depending on n) bilinear form on V_{n-1}. For $m = 0, 1, \ldots, n-1$ define $C_m \subset V_{n-1}$ to be a two-dimensional cone of order m, given by all polynomials $\mathbf{t} = \mathbf{p}^m \mathbf{r}$, where $\mathbf{p} \in V_1$, and $\mathbf{r} \in V_{n-m-1}$.

Proof of Theorem 1.2. We want to define a paraconformal structure by requiring

$$\sigma^a_{A_1 \cdots A_{n-1}} \frac{\partial Z}{\partial t^a} = p_{A_1} \cdots p_{A_n},$$

where $y = Z(x, t)$ is a surface in M corresponding to a solution of (1.3). The symbols σ are of course independent of x, but the p_As (and the corresponding polynomials) do depend on x as does Z.

We shall first assume that this paraconformal structure exists on M and establish the conditions satisfied by the ODE.

All sections of $\mathbb{S}$ correspond to degree one homogeneous polynomials in ζ^A. If $(p_0, p_1) \in \mathbb{R}^2$, then $\mathbf{p} = p_0 \zeta^0 + p_1 \zeta^1 \in V_1$. Let $\mathbf{p} \in V_1$, and let $\mathbf{q} = \mathbf{p}'$. The fibres of the spin bundle are two-dimensional, therefore

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}x} = P\mathbf{p} + Q\mathbf{q} \tag{2.2}$$

for some $P, Q \in V_0$. These relations hold on the correspondence space (1.4) and $\mathbf{p}$, $\mathbf{q}$ are regarded as x-dependent sections of the bundle $\mathbb{S}$ pulled back from M to $\mathcal{F}$.

Consider an element T of C_{n-1} (a maximally null vector). The maximally null vectors correspond to polynomials $\mathbf{T} = \mathbf{p}^{n-1}$ with a repeated root of multiplicity $(n-1)$. Each solution $y = Z(x, t)$ defines a section of T^*M given by the gradient on M:

$$E = \nabla Z.$$

Assume that E is a maximally null one-form (an element of C_{n-1}), and construct a frame of n one-forms on T^*M given by

$$E, E', E'', \dots, E^{(n-1)}.$$

Requiring that these forms are linearly independent imposes a condition of non-degeneracy, which is an open condition on Ψ in relation (1.1).

The corresponding polynomials are of the form:

$$\begin{aligned}
&\mathbf{E} = \mathbf{p}^{n-1}\\
&\mathbf{E}' = 0 + a_{11}\mathbf{p}^{n-2}\mathbf{q}\\
&\mathbf{E}'' = a_{20}\mathbf{p}^{n-1} + a_{21}\mathbf{p}^{n-2}\mathbf{q} + a_{22}\mathbf{p}^{(n-3)}\mathbf{q}^2\\
&.\\
&.\\
&.\\
&\mathbf{E}^{(n-1)} = a_{(n-1)0}\mathbf{p}^{n-1} + a_{(n-1)1}\mathbf{p}^{n-2}\mathbf{q} + \cdots + a_{(n-1)(n-1)}\mathbf{q}^{n-1},
\end{aligned}$$

or in general

$$\mathbf{E}^{(i)} = \sum_{k=0}^{i} a_{ik}\mathbf{p}^{n-1-k}\mathbf{q}^k, \quad i = 0, \dots, n-1. \tag{2.3}$$

The upper triangular matrix (a_{ij}) can be computed using (2.2). It depends on P, Q and their derivatives with respect to x. The nth derivative of $\mathbf{E}$ with respect to x is given by

$$\mathbf{E}^{(n)} = a_{n0}\mathbf{p}^{n-1} + a_{n1}\mathbf{p}^{n-2}\mathbf{q} + \cdots + a_{n(n-1)}\mathbf{q}^{n-1}.$$

Remembering that $y = Z(x, t)$ is a solution to (1.3), and using the chain rule we express $\mathbf{E}^{(n)}$ as a linear combination of $\mathbf{E}, \mathbf{E}', \dots, \mathbf{E}^{(n-1)}$ by

$$\mathbf{E}^{(n)} = \sum_{i=0}^{n-1} F_i \mathbf{E}^{(i)}. \tag{2.4}$$

This gives rise to

$$a_{nj} = \sum_{i=0}^{n-1} F_i a_{ij}. \tag{2.5}$$

Solving these n equations for P and Q yields:

$$Q = \frac{2}{n(n-1)} F_{n-1},$$
$$P = \frac{1}{n(n^2-1)} \left(\frac{(3n-1)(n-2)}{n(n-1)} F_{n-1}^2 + 6F_{n-2} - 2(n-2)\frac{\mathrm{d}}{\mathrm{d}x} F_{n-1} \right) \tag{2.6}$$

(the calculations leading to these formulae are presented in Appendix A). The remaining equations imply the vanishing of $(n-2)$ expressions constructed out of F. Each expression is a polynomial C_k in the derivatives of F of order less than or equal to n, of the form (1.10). The vanishing of these expressions characterises a class of ODEs (1.3) such that their solution spaces admit a paraconformal structure (1.9). It follows from the construction (and it may be checked if desired) that vanishing of these expressions is invariant under point transformations:

$$\hat{x} = \hat{x}(x, y), \quad \hat{y} = \hat{y}(x, y).$$

Conversely, let us assume that we are given an ODE (1.3) such that conditions (1.10) hold. We define P and Q by (2.6) and solve the linear system of ODEs $\mathbf{q} = \mathbf{p}'$, $\mathbf{q}' = P\mathbf{p} + Q\mathbf{q}$ to determine $\mathbf{p}$ and $\mathbf{q}$. We then define a basis of TM by the procedure of taking gradients leading to (2.3). The consistency conditions are guaranteed by the calculation in the first part of the proof. This gives a paraconformal structure such that the surfaces $y = Z(x, t)$ in M are totally null. □

If $n = 2$ the paraconformal structure always exists. If $n = 3$ condition (1.10) is given by (1.7). If $n = 4$ we have two conditions given by (1.11) and (1.12). The polynomials corresponding to $n = 5$ are given in Appendix A.

There are no terms of the form F_{ij} (recall that $F_{ij} = \partial^2 F/\partial y^{(i)}\partial y^{(j)}$) in C_k, so the coefficients of the polynomials C_k are determined by looking at the special case of linear homogeneous equations, where

$$F = p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y,$$

and $F_k = p_k(x)$.

2.1. Example

The general solution of (2.5) regarded as an overdetermined system of PDEs for F appears to be intractable. To find some examples we seek $F = F(y^{(n-1)}, x)$. Let $z := y^{(n-1)}$. ODE (1.3) reduces to a first-order ODE, and a sequence of quadratures:

$$z' = F(z, x), \qquad y(x) = \int^{x}\int^{x_{n-1}}\cdots\int^{x_2} z(x_1)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}.$$

We use MAPLE to verify that all constraint equations (1.10) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{n-1} = \frac{1}{n}(F_{n-1})^2, \quad \text{where now } \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + F\frac{\partial}{\partial z}.$$

We therefore need to solve

$$\frac{\partial^2 F}{\partial z\partial x} = \frac{1}{n}\left(\frac{\partial F}{\partial z}\right)^2 - F\frac{\partial^2 F}{\partial z^2}.$$

The Legendre transformation $s = F_z$, $G(s, x) = F(z(s, x), x) - sz(s, x)$ gives a linear equation for G. This transform can then be inverted, and the solution can be found for an arbitrary 'initial data' $F(z, 0)$. To write down an explicit example make a further assumption that F is independent of x, which yields:

$$F(z) = (az + b)^{n/(n-1)}.$$

Redefining $y(x)$ by a point transformation we can set $b = 0$, so the nth-order ODE is

$$y^{(n)} = (ay^{(n-1)})^{n/(n-1)}. \tag{2.7}$$

The corresponding n-parameter family of solutions to (1.3) is readily found

$$y = t^1 + t^2 x + \cdots + t^{n-1}x^{n-2} - \frac{(n-1)^{n-1}}{a^n(n-2)!}\ln(x + t^n).$$

If $n = 3$ the space of solutions to (1.3) is equipped with the NIL Einstein–Weyl structure [18] (see also Section 7).

3. Comparison with Doubrov–Wilczynski invariants

In this section we shall demonstrate that for a given ODE the generalised Wünschmann conditions (1.10) are equivalent to the vanishing of a set of invariants constructed by Doubrov [9]. Doubrov's work builds on an old theorem of Wilczynski [19], which we shall review first.

Let

$$y^{(n)} = p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y \tag{3.1}$$

be a linear homogeneous nth-order ODE, defined up to transformations of the form:

$$(y, x) \longrightarrow (a(x)y, b(x)).$$

Wilczynski has demonstrated that ODE (3.1) is trivialisable by this transformation if $(n - 2)$ expressions constructed out of the $p_i(x)$s and their derivatives vanish. More formally he has shown the following theorem.

Theorem 3.1 (Wilczynski [19]). *The following three conditions are equivalent*:

(a) *The Wilczynski invariants*:

$$\theta_k\left(p_i, \frac{\mathrm{d}p_i}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}p_i}{\mathrm{d}x^{n-1}}\right) = 0, \quad i = 0, \ldots, n-1,\ k = 1, \ldots, n-2. \tag{3.2}$$

of the equation (3.1) *vanish*.

(b) *The equation* (3.1) *can be transformed to* $y^{(n)} = 0$ *by a change of variables* $(y, x) \to (a(x)y, b(x))$.

(c) *Let* $y_1(x), \ldots, y_n(x)$ *be any basis of the solution space of* (3.1). *Then the embedding* $\mathbb{R}P^1 \to \mathbb{R}P^{n-1}$ *given by* $x \to [y_1(x) : \cdots : y_n(x)] \subset \mathbb{R}P^{n-1}$, *is an open subset of the normal rational curve*.

The expressions θ_k are polynomials in p_is and their derivatives of order less than or equal to n. Wilczynski has produced compact formulae for these polynomials if $p_{n-1}(x) = p_{n-2}(x) = 0$ (it is always possible to set these two coefficients to zero by a choice of $a(x), b(x)$). We shall not need these formulae in what follows.

The Doubrov invariants introduced in [9] for general ODEs are the classical Wilczynski invariants (3.2) of their linearizations. That is, when restricted to any solution (1.3), they produce the classical Wilczynski invariants of the linearization of (1.3) at this solution. In practise, Doubrov's invariants are calculated by replacing $p_k(x)$ by $F_k(x, y, \ldots, y^{(n-1)})$ in Wilczynski's invariants.

Theorem 3.2 (Doubrov [9]). *Let* θ_k *be the Wilczynski invariants of a linear nth-order homogeneous ODE. Then the vanishing of all the expressions*:

$$L_k = \theta_k\left(F_i, \frac{\mathrm{d}F_i}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}F_i}{\mathrm{d}x^{n-1}}\right), \quad i = 0, \ldots, n-1,\ k = 1, \ldots, n-2, \tag{3.3}$$

is invariant under contact transformations of the general nth-order ODE (1.3).

The invariants L_k will be from now on called the Doubrov invariants.

The ideas behind the proof of the next theorem are due to Doubrov. The geometric content of this theorem (constraining a curve until it becomes a rational normal curve) was also known to Bryant [4].

Theorem 3.3. *The vanishing of the generalised Wünschmann conditions (1.10) is equivalent to the vanishing of the Doubrov invariants (3.3).*

To prove this theorem we shall need the following lemma.

Lemma 3.4. *The existence of the paraconformal structure $TM = \mathbb{S} \odot \mathbb{S} \odot \ldots \odot \mathbb{S}$ is equivalent to the existence of a sub-bundle of $\mathbb{P}(TM)$ with rational normal curves as fibres.*

Proof. For any rational normal curve $\mathcal{C} \subset \mathbb{P}(V)$, where V is an n-dimensional vector space there exists an isomorphism $V \to V_{n-1}$ which identifies $\mathcal{C}$ with $x \to (1, x, \ldots, x^{n-1})$. Any two such isomorphisms differ by a $PGL(2, \mathbb{R})$ projective transformation.

Now if M is paraconformal then $T_t M$ is isomorphic to V_{n-1} for all $t \in M$. Any element $\mathbf{p} \in \mathbb{S}$ (a two dimensional spinor) gives a totally null cone $\mathbf{p}^{n-1} \in T_t M$, or a rational normal $\mathcal{C}_t$ in $P(T_t M)$. The union of $\mathcal{C}_t$ as t varies in M gives a sub-bundle of $\mathbb{P}(TM)$. $\square$

Proof of Theorem 3.3. If the Doubrov invariants (3.3) vanish, then any linearisation of (1.3) has vanishing Wilczynski invariants, and by Theorem 3.1 we have a rational normal curve in the projectivization of each tangent space to the solution space. This, by Lemma 3.4, is equivalent to the existence of the paraconformal structure on the solution space. Theorem 1.2 now implies that conditions (1.10) are satisfied.

Conversely, let ODE (1.3) satisfy (1.10), and let $y = Z(x, t)$, $t = (t^1, \ldots, t^n)$ be its general solution. Fix $t = T_0$. The vector $E = \partial_a y$ in $T_{T_0} M$, where $\partial_a = \partial/\partial t^a$, is maximally null with respect to the paraconformal structure defined by the ODE, so that it is given by $\mathbf{p}^{n-1}$. Each component of $E = \nabla Z$ is a derivative of a solution to ODE (1.3), and as such it satisfies the linearisation of (1.3) around $Y = Z(x, T_0)$, as the differentiation of (1.3) yields:

$$\frac{\partial y^{(n)}}{\partial t^a} = \sum_{i=0}^{n-1} \left(\frac{\partial F}{\partial y^{(i)}} |_{y=Y} \right) \frac{\partial y^i}{\partial t^a}. \tag{3.4}$$

These equations are homogeneous linear ODEs for $\partial y/\partial t^a$. But then the nullity of E implies that the embedding $x \to [\partial_1 y(x) : \cdots : \partial_n y(x)]$ is a rational normal curve for any T_0, and so the linearised ODE (3.4) is trivialisable by Wilczynski's Theorem 3.1. Therefore, ODE (1.3) has vanishing Doubrov invariants (3.3) which proves the relative equivalence of the two sets of invariants (i.e. their vanishing is equivalent). $\square$

In Section 6 we shall need another result of Doubrov's.

Theorem 3.5 (Doubrov [9]). *ODE (1.3) is trivialisable by contact transformations if and only if invariants (3.3) vanish, and the following conditions hold*:

- $n = 4, \quad F_{333} = 6F_{233} + F_{33}^2 = 0,$
- $n = 5, \quad F_{44} = 6F_{234} - 4F_{333} - 3F_{34}^2 = 0,$
- $n = 6, \quad F_{55} = F_{45} = 0,$
- $n \geq 7, \quad F_{(n-1)(n-1)} = F_{(n-1)(n-2)} = F_{(n-2)(n-2)} = 0.$

4. Twistor theory

We shall now show how, in the real analytic case, the paraconformal structure (1.9) on M can be encoded in a holomorphic geometry of rational curves embedded in a complex surface $\mathcal{Z}$. The nth-order ODE on $\mathcal{Z}$ will then implicitly be given by the embedding $L \subset \mathcal{Z}$, provided that L has self-intersection number n.

In this section we regard (1.1) as a holomorphic relation between complex coordinates (x, y, t), and examine the geometry of the complexified correspondence space (1.4) and the associated double fibration picture. The surface $\mathcal{Z}$ will eventually become a twistor space of (1.3). We shall however set up a more general correspondence, and consider Legendrian curves in three-dimensional twistor spaces.

Let Y be a complex three-fold with an embedded rational curve L with a normal bundle $N = \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)$. We have $H^1(\mathbb{C}P^1, \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)) = 0$, and so the moduli space of such curves in Y is a manifold $\mathcal{M}$ of dimension equal to

$$\dim H^0(\mathbb{C}P^1, \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)) = 2n-2.$$

Now we restrict our attention to a moduli space M of contact (Legendrian) curves with normal bundle N. The canonical line bundle of holomorphic three-forms on Y restricted to a curve L is

$$\kappa(Y) = T^*L \otimes \Lambda^2(N^*) = \mathcal{O}(2-2n),$$

since $T^*\mathbb{C}P^1 = \mathcal{O}(-2)$. From the general theory of contact structures it follows that the contact line bundle is given by ${L_c}^2 = \kappa(Y)$. Now pick a section of ${L_c}^*$ (a contact one-form), and contract it with a tangent vector to a rational curve to get a section of

$$(\mathcal{O}(1-n) \otimes \mathcal{O}(2))^*.$$

The vanishing of this section (the Legendrian condition) gives $\dim H^0(\mathbb{C}P^1, \mathcal{O}(n-3)) = n-2$ conditions on $\mathcal{M}$. Therefore, the dimension of the moduli space M of Legendrian curves is

$$\dim M = (2n-2) - (n-2) = n.$$

This can be summarised by the double fibration picture:

$$M \xleftarrow{\hat{p}} \hat{\mathcal{F}} \xrightarrow{\hat{q}} Y. \tag{4.1}$$

The curves $\hat{q}(\hat{p}^{-1}(\mathbf{t})) \cong \mathbb{C}P^1$ are Legendrian with respect to the contact form on Y. Bryant's generalisation [3] of the Kodaira theorems guarantees that the moduli space M of Legendrian rational curves is stable under small deformations of Y.

Consider the special case $Y = \mathbb{P}(T\mathcal{Z})$. A rational curve L with normal bundle $\mathcal{O}(n-1)$ in $\mathcal{Z}$ has a natural lift $\hat{L}$ to Y, given by $z \in L \to (z, \dot{z} \in T_z L)$. The lifted curves are Legendrian with respect to the canonical contact structure on the projectivised tangent bundle. The double fibration (4.1) reduces to (1.5).

The existence of the complexified paraconformal structure (1.9) follows from the structure of the normal bundle. From Kodaira [15] theory, since the appropriate obstruction groups vanish, we have

$$T_t M = \Gamma(L_t, N_t) = S^{n-1}(\mathbb{S}_t), \quad \mathbb{S}_t = \mathbb{C}^2, \tag{4.2}$$

where N_t is the normal bundle to the rational curve $L_t = \mathbb{C}P^1$ in $\mathcal{Z}$ corresponding to the point $t \in M$. The nontrivial examples of ODEs satisfying all the constraints (1.10) can therefore be constructed by applying algebraic operations on a rational curve embedded in a total space of $\mathcal{O}(N)$ for N sufficiently large [12].

The correspondence space $\mathcal{F} = M \times \mathbb{C}P^1$ is equipped with a canonical $(n-1)$ dimensional distribution $\mathcal{D}$, such that $\mathcal{Z} = \mathcal{F}/\mathcal{D}$. The normal bundle to a rational curve $L_t := q(p^{-1}(\mathbf{t}))$ consists of vectors tangent to M at t (horizontally lifted to $T_{t,\lambda}\mathcal{F}$) modulo $\mathcal{D}$. Therefore, we have a sequence of sheaves over $\mathbb{C}P^1$:

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathbb{C}^n \longrightarrow \mathcal{O}(n-1) \longrightarrow 0.$$

The map $\mathbb{C}^n \longrightarrow \mathcal{O}(n-1)$ is given by $V^{A_1A_2\cdots A_{n-1}} \longrightarrow V^{A_1A_2\cdots A_{n-1}} z_{A_1} z_{A_2} \cdots z_{A_{n-1}}$. Its kernel consists of vectors of the form $z^{(A_1}\lambda^{A_2\cdots A_{n-1})}$ with $\lambda^{A_2\cdots A_{n-1}} \in \mathbb{C}^{n-1}$ varying. The twistor distribution is therefore $\mathcal{D} = \mathcal{O}(-1) \otimes S^{(n-2)}(\mathbb{C}^2)$. This distribution is the geodesic spray (6.3) if $n = 2$, or the Einstein–Weyl Lax pair (7.1) if $n = 3$.

5. Exotic $\mathcal{G}_3$ holonomy and fourth-order ODEs

In this Section we shall make contact with Bryant's work [3] and show that if $n = 4$ the Wünschmann conditions (1.10) are equivalent to the existence of certain exotic holonomy on M.

Recall the notation introduced at the beginning of Section (2) and define $\mathcal{G}_k \subset GL(V_k)$ by

$$\mathcal{G}_k = \{g \in GL(V_k) | g(\mathbf{t}) \in C_k \text{ if } \mathbf{t} \in C_k\}.$$

Can $\mathcal{G}_{n-1}$ appear as a holonomy group of a torsion-free connection of an n-dimensional manifold? Bryant [3] has examined Berger's criteria, and established that the answer is 'no' if $n > 5$ (the case $n = 5$ is special, as the five-dimensional representation of $SL(2, \mathbb{C})$ is the holonomy of a symmetric space $M = SL(3, \mathbb{C})/SL(2, \mathbb{C})$).

Let $\langle , \rangle_2$ be given by (2.1), and let

$$g(X, X, X, X) := \langle \langle X, X\rangle_2, \langle X, X\rangle_2\rangle_2$$

be a $\mathcal{G}_3$ invariant quartic form on TM.

Definition 5.1. The vector $X \in T_tM$ is null iff

$$Q(X) := g(X, X, X, X) = 0.$$

The α-plane is a two-dimensional plane in T_tM spanned by vectors X, Y such that

$$Q(X + \lambda Y) = 0$$

for each value of a parameter λ.

The null vectors in this sense correspond to polynomials of the form $\mathbf{p}^2\mathbf{r}$. Vectors in an α-plane are then obtained by varying $\mathbf{r}$ and keeping $\mathbf{p}$ fixed.

Theorem 5.2 (Bryant [3]). *A four-dimensional manifold M admits a torsion free connection with holonomy $\mathcal{G}_3$ iff for every α-plane there exists a two-dimensional surface $\Sigma \subset M$ (called α-surface) tangent to this α-plane.*

The space of torsion-free $\mathcal{G}_3$ structures modulo diffeomorphisms depends on four arbitrary functions of three variables.

Bryant has also shown [3] that if M admits a $\mathcal{G}_3$ structure, then there exists a three-parameter family of α-surfaces. In the complexified category this family is parametrised by points of a complex three-fold Y, and the Legendrian $\mathcal{O}(2) \oplus \mathcal{O}(2)$ curves in Y correspond to points in M (compare this with the twistorial treatment in Section 4).

Proof of Theorem 1.3. Conditions (1.11) and (1.12) are the invariants (1.10) with $n = 4$ which establishes the equivalence of (1) and (3).

To show (1) → (2) observe that the existence of the paraconformal structure (1.9) implies the existence of a symplectic structure ε up to scale on each spin space. This gives us the symmetric quartic form

$$g(X, X, X, X) = \varepsilon^{A_2B_2}\varepsilon^{A_3B_3}\varepsilon^{C_2D_2}\varepsilon^{C_3D_3}\varepsilon^{A_1C_1}\varepsilon^{B_1D_1}X_{A_1A_2A_3}X_{B_1B_2B_3}X_{C_1C_2C_3}X_{D_1D_2D_3}.$$

The quartic $Q(X) = 0$ selects null vectors, and α-planes. Theorem 5.2 asserts that these planes are integrable iff they come from a $\mathcal{G}_3$ structure. But they will always be integrable in the paraconformal case because of the following interpretation. Fixing a point in $Z \in \mathcal{Z}$ gives a three-dimensional surface in $N \subset M$, such that its normal ∇Z is a perfect cube. Fixing a point and a direction in $\mathcal{Z}$ gives an α-surface $\Sigma \subset N$ (think of Z, Z', Z'', Z''' as coordinates in M). This corresponds to fixing a point in $Y = \mathbb{P}(T^*\mathcal{Z})$, and to Bryant's α-plane with normals Z, Z' which are gradients (so that it really is a surface).

It remains to demonstrate (2) → (1). Suppose we look directly for the quartic as

$$g = g_0 + g_1, \tag{5.1}$$

where g_0 is the form when the ODE is trivial ($F = 0$) and g_1 is a combination of all possible lower order terms (lower order in the sense of fewer derivatives of y):

$$\begin{aligned} g_0 &= 18\,\mathrm{d}y\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r - 9(\mathrm{d}y)^2(\mathrm{d}r)^2 + 3(\mathrm{d}p)^2(\mathrm{d}q)^2 - 8(\mathrm{d}p)^3\,\mathrm{d}r - 6\,\mathrm{d}y(\mathrm{d}q)^3, \\ g_1 &= \alpha(\mathrm{d}y)^4 + \beta(\mathrm{d}y)^3\,\mathrm{d}p + \gamma(\mathrm{d}y)^2(\mathrm{d}p)^2 + \delta(\mathrm{d}y)^3(\mathrm{d}q) + \epsilon\,\mathrm{d}y(\mathrm{d}p)^3 + \xi(\mathrm{d}y)^2\,\mathrm{d}p\,\mathrm{d}q \\ &\quad + \eta(\mathrm{d}y)^3\,\mathrm{d}r + \kappa(\mathrm{d}p)^4 + \gamma(\mathrm{d}y)(\mathrm{d}p)^2\,\mathrm{d}q + \mu(\mathrm{d}y)^2(\mathrm{d}q)^2 + \nu(\mathrm{d}y)^2\,\mathrm{d}p\,\mathrm{d}r + \zeta(\mathrm{d}p)^3\,\mathrm{d}q \\ &\quad + \pi\,\mathrm{d}y\,\mathrm{d}p(\mathrm{d}q)^2 + \theta\,\mathrm{d}y(\mathrm{d}p)^2\,\mathrm{d}r + \phi(\mathrm{d}y)^2\,\mathrm{d}q\,\mathrm{d}r, \end{aligned}$$

and $p = y', q = y'', r = y''', s = y'''' = F(x, y, p, q, r)$.

We need to fix 15 coefficients $(\alpha, \ldots, \phi)$. We impose

$$g' = \Lambda g$$

for some Λ and work systematically through the coefficients, fixing them in order (see Appendix A for the details of this calculation). We solve Eqs. (A.1)–(A.10), (A.11) and (A.13). Now equation (A.12) becomes (1.12). Then we solve (A.14) and (A.16). Now (A.15) and (1.12) give (1.11).

The remaining conditions (A.17)–(A.20) give two more conditions on F, but we know these will be satisfied because of the paraconformal argument: once we impose (1.11) and (1.12) the quartic will exist and we can check that the coefficients we have found by a direct approach agree with what the paraconformal method has told us. Thus, (1.11) and (1.12) are necessary and sufficient for integrability. □

Readers who compare our treatment with that of Bryant's will recall (Theorem 4.5 in [3]) that the existence of a torsion-free $\mathcal{G}_3$-structure on the moduli space was equivalent to the vanishing of two primary invariants (A, C), and two secondary invariants (B, D) which are well defined only if $A = C = 0$. All these invariants are polynomials in the third-order derivatives of the function F. Bryant has pointed out [4] that two conditions (1.11) and (1.12) are equivalent to the vanishing of A and B. If $A = B = 0$ then C and D vanish identically.

6. Torsion-free paraconformal connections

Let M admit a paraconformal structure, and let

$$D : \Gamma(\mathbb{S}^k) \longrightarrow \Gamma(\mathbb{S}^k \otimes T^*M) = \Gamma(\mathbb{S}^{k+n-1}), \quad \text{where } \mathbb{S}^k = \mathbb{S}^{\otimes k},\ k = 0, 1, \dots, n-1$$

be a connection. In this section we shall show that if D preserves the paraconformal structure (1.9) then it necessarily has torsion if $n = 4$ or $n \geq 6$.

Definition 6.1. The connection D is called paraconformal if its action on elements of V_1 is given by

$$D\mathbf{p} = U \otimes \mathbf{p} + V \otimes \mathbf{q}$$

for some $U, V \in \mathbb{S}^{n-1}$.

This definition implies that $D\mathbf{E} = (n-1)U \otimes \mathbf{E} + V \otimes \mathbf{E}'$. The torsion free condition becomes

$$U = A\mathbf{E} + B\mathbf{E}', \quad V = (n-1)B\mathbf{E} + C\mathbf{E}'$$

for some $A, B, C \in V_0$. Demanding that the connection does not depend on (x, y) gives the consistency condition:

$$(D\mathbf{p})'' = D(\mathbf{p}''), \tag{6.1}$$

which yields

$$0 = (D\mathbf{p})'' - D(P\mathbf{p} + Q\mathbf{q}) = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - Q\frac{\mathrm{d}}{\mathrm{d}x} - P\right) D\mathbf{p} - (DP) \otimes \mathbf{p} - (DQ) \otimes \mathbf{q}$$

$$= (\alpha_1\mathbf{E} + \alpha_2\mathbf{E}' + \alpha_3\mathbf{E}'' + \alpha_4\mathbf{E}''') \otimes \mathbf{p} + (\beta_1\mathbf{E} + \beta_2\mathbf{E}' + \beta_3\mathbf{E}'' + \beta_4\mathbf{E}''') \otimes \mathbf{q}$$
$$- \left(\sum_{i=0}^{n-1} \mathbf{E}^{(i)} \frac{\partial P}{\partial y^{(i)}} \otimes \mathbf{p} + \sum_{i=0}^{n-1} \mathbf{E}^{(i)} \frac{\partial Q}{\partial y^{(i)}} \otimes \mathbf{q}\right).$$

The coefficients of $\mathbf{E}^{(i)} \otimes \mathbf{p}$, and $\mathbf{E}^{(i)} \otimes \mathbf{q}$ have to vanish, which gives $2n$ conditions on F. Here (P, Q) are given by (2.6) and $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4$ can be determined in terms of A, B, C, P, Q, and their derivatives:

- $n = 2$. In this case $\mathbf{E}''$ and $\mathbf{E}'''$ are determined in terms of $\mathbf{E}$ and $\mathbf{E}'$ according to (2.4), and therefore $\alpha_3, \alpha_4, \beta_3, \beta_4$ all vanish. There are no conditions on F arising from (6.1). However, imposing the geodesic conditions on curves $Z = \text{const}$ yields (1.6). Let t^A and $z^A = \dot{t}^A$ be local coordinates on M and T_tM. The Christoffel symbols $\Gamma_{AB}^C = \Gamma_{AB}^C(t)$ of D are defined up to projective equivalence:

  $$\Gamma_{AB}^C \sim \Gamma_{AB}^C + \delta_{(A}^C\omega_{B)}$$

  for some $\omega_B = \omega_B(t)$. Let $t^C = t^C(\tau)$ be solutions to

  $$\ddot{t}^C + \Gamma_{AB}^C\dot{t}^A\dot{t}^B = v\dot{t}^C, \qquad \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{6.2}$$

where v is some function. These geodesic curves lift to the integral curves of the geodesic spray which is a projection of

$$L = z^A \frac{\partial}{\partial t^A} - \Gamma^C_{AB} z^A z^B \frac{\partial}{\partial z^C} \tag{6.3}$$

from TM to $\mathbb{P}(TM)$. Eliminating τ from (6.2) leads to a second-order ODE for $t^1 = t^1(t^0)$ which is at most cubic in the first derivatives (the cubic term is given by $\varepsilon_{CD}\Gamma^C_{AB} z^A z^B z^D$, where z^A are homogeneous coordinates on $\mathbb{P}(TM)$, and $z^1/z^0 = \mathrm{d}t^1/\mathrm{d}t^0$). This ODE is dual to (1.3) in the sense of Cartan [5]. It could also be read off from relation (1.1) by rewriting it as $t^1 = K(t^0, x, y)$, and eliminating (x, y) between K and its first two derivatives w.r.t. t^0. A second-order ODE (1.3) is trivialisable by point transformations iff the curvature of the projective connection vanishes. This curvature vanishes iff (1.6) holds, and $F_{1111} = 0$.

- $n = 3$. Now α_4 and β_4 vanish. The compatibility conditions in (6.1) fix A, B, C. Imposing the totally geodesic condition $C = 0$ on the null surfaces $Z = \text{const}$ gives the constraint (1.8). We shall come back to this case in the next section.
- $n \geq 4$. The coefficients of $\mathbf{E}''' \otimes \mathbf{p}$, $\mathbf{E}''' \otimes \mathbf{q}$ and $\mathbf{E}'' \otimes \mathbf{p}$ fix (A, B, C) in terms of F and its derivatives and the coefficients of $\mathbf{E}'' \otimes \mathbf{q}$, $\mathbf{E}'$ and $\mathbf{E}$ give five equations (A.21)–(A.25) for F (the details are in Appendix A). In particular (A.23) yields:

$$\left(\frac{6n-8}{n}\right) F_{n-1} F_{(n-1)3} + (8-2n)\frac{\mathrm{d}}{\mathrm{d}x} F_{(n-1)3} - (2n-2) F_{(n-1)2} + 6F_{(n-2)3} = 0. \tag{6.4}$$

Proof of Theorem 1.4. If $n = 4$, condition (6.4) reduces to

$$F_{33} = 0,$$

and Wünschmann conditions (1.11) and (1.12) now imply $A = B = C = 0$. Some computer algebra reduces (1.11) and (1.12) to

$$F = \alpha(x) + \beta(x)y + \gamma(x)y' + \delta(x)y'' + \epsilon(x)y''',$$

where $\alpha(x), \delta(x), \epsilon(x)$ are arbitrary, and

$$\beta = \frac{11}{1600}\epsilon^4 - \frac{9}{50}\epsilon^2\epsilon' - \frac{1}{200}\epsilon^2\delta + \frac{21}{100}(\epsilon')^2 + \frac{1}{50}\epsilon'\delta - \frac{9}{100}\delta^2 + \frac{7}{20}\epsilon\delta'' - \frac{1}{5}\epsilon'''$$
$$+ \frac{3}{10}\delta'' - \frac{1}{4}\epsilon\delta', \qquad \gamma = \frac{3}{4}\epsilon\epsilon' - \frac{1}{2}\epsilon'' + \delta' - \frac{1}{2}\delta\epsilon - \frac{1}{8}\epsilon^3.$$

We can however perform a point transformation (which is in fact fibre preserving):

$$y = a(x)\hat{y}(x) + b(x), \qquad x = c(\hat{x}),$$

and choose the functions (a, b, c) to set $\alpha = \delta = \epsilon = 0$. The resulting fourth-order ODE (1.3) is therefore trivial up to point transformations.

If $n > 4$, then the coefficients of $\mathbf{E}^{(k)}$ with $k > 3$ give

$$\frac{\partial P}{\partial y^{(k)}} = \frac{\partial Q}{\partial y^{(k)}} = 0, \quad k = 4, 5, \ldots, n-1. \tag{6.5}$$

Conditions (6.5) give the following:

1. $n = 6$:

$$F_{55} = F_{45} = 3F_{44} - 8F_{53} = 0, \tag{6.6}$$

2. $n \geq 7$:

$$F_{(n-1)k} = F_{(n-2)(k+1)} = 3F_{(n-2)4} - (n-2)F_{(n-1)3} = 0, \quad k = 4, \ldots, n-1. \tag{6.7}$$

Eqs. (1.10), (A.21)–(A.25) also have to be satisfied. Eqs. (6.6) and (6.7) and Theorem 3.5 imply that ODE (1.3) is trivialisable by contact transformation if $n \geq 6$. □

The case $n = 5$ is exceptional. Conditions (6.5) give

$$F_{44} = 0, \tag{6.8}$$

but this is not sufficient to guarantee the trivialisability. In fact the five parameter family of conics in the complex projective plane gives a counterexample. In this case M (a real form of $PSL(3, \mathbb{C})/SO(3, \mathbb{C})$) is a five-dimensional space with the paraconformal structure which admits a torsion free paraconformal connection. The five-parameter family of conics in $\mathbb{C}P^2$ is given by (1.1) with

$$\Psi = y^2 + t^1 x^2 + 2t^2 xy + t^3 x + t^4 y - t^5,$$

where (y, x) are inhomogeneous coordinates on $\mathbb{C}P^2$. We regard y as a function of x, and implicitly differentiate relation (1.1) five times w.r.t. x. Solving for $y^{(5)}$ yields the desired fifth-order ODE:

$$y^{(5)} = -\frac{40r^3}{9q^2} + 5\frac{rs}{q}, \tag{6.9}$$

where $y' = p$, $y'' = q$, $y''' = r$, $y'''' = s$.

Wünschmann conditions (1.11) and (1.12), as well as the six conditions (6.5), (A.21)–(A.25) hold, so the paraconformal torsion free connection exists in this case. Eq. (6.9) is nevertheless not contact equivalent to $y^{(5)} = 0$, as the invariant $6F_{234} - 4F_{333} - 3F_{34}^2$ from Theorem 3.5 does not vanish, and is equal to $(5/3)q^{-2}$.

7. From Einstein–Weyl structures to third-order ODEs

In three dimensions the existence a paraconformal structure (1.9) is equivalent to the existence of a conformal structure $[h]$ of signature $(++-)$. This is a well known fact based on representing vectors as symmetric matrices:

$$X^a = (X^1, X^2, X^3) \longrightarrow X^{AB} = \begin{pmatrix} X^1 + X^2 & X^3 \\ X^3 & X^1 - X^2 \end{pmatrix} \in \Gamma(\mathbb{S} \otimes \mathbb{S}),$$

where X^a are components of X w.r.t. some basis.

The matrices corresponding to null vectors (i.e. $h(X, X) = 0$, $h \in [h]$) have vanishing determinant, and must have rank one. Therefore, $X^{AB} = p^A p^B$ for such vectors.

Set $n = 3$, and assume that the third-order ODE (1.3) satisfies Wünschmann condition (1.7). For each choice of (x, y) (1.1) defines a surface in M which is null w.r.t. $[h]$. In the last section we have shown that if $n = 3$ the null surfaces $y = Z(x, t^a)$ are totally geodesic w.r.t. some torsion-free connection D if $F(x, y, y', y'')$ satisfies the constraint (1.8), and it is well known [7,14,17] that the Einstein–Weyl (EW) equations are equivalent to the existence of a two-dimensional family of surfaces $\Sigma \subset M$ which are null with respect to $[h]$, and totally geodesic with respect to D.

Let M be a three-dimensional manifold with a torsion-free connection D, and a conformal structure $[h]$ of signature $(++-)$ which is compatible with D in a sense $Dh = \omega \otimes h$ for some one-form ω. Here $h \in [h]$ is a representative metric in a conformal class. If we change this representative by $h \to \psi^2 h$, then $\omega \to \omega + 2\mathrm{d}\ln\psi$, where ψ is a non-vanishing function on W. A triple $(M, [h], D)$ is called a Weyl structure. The conformally invariant Einstein–Weyl (EW) equations state that the symmetrised part of the Ricci tensor of D is proportional to the representative of $[h]$.

Given a third-order ODE which satisfies (1.7) and (1.8), the EW structure can be reconstructed following the steps described in [7,18]. The problem of reconstructing the ODE starting from a given EW structure was left open in these references. We shall present a method which reduces the problem of finding the allowed ODE to a system of linear PDEs. First recall the Lax representation for the EW equations [11]. Let X_1, X_2, X_3 be three-independent vector fields on M, and let e_1, e_2, e_3 be the dual one-forms. Assume that

$$h = e_2 \otimes e_2 - 2(e_1 \otimes e_3 + e_3 \otimes e_1)$$

and some one-form ω together give an EW structure. Let $X(\lambda) = X_1 - 2\lambda X_2 + \lambda^2 X_3$ where $\lambda \in \mathbb{C}P^1$ is a projective coordinate on the fibres of $\mathbb{S} \to M$. Then $h(X(\lambda), X(\lambda)) = 0$ for all $\lambda \in \mathbb{C}P^1$ so $X(\lambda)$ determines a sphere of null vectors. The vectors $X_1 - \lambda X_2$ and $X_2 - \lambda X_3$ form a basis of the orthogonal complement of $X(\lambda)$. For each $\lambda \in \mathbb{C}P^1$ they span a null two-surface. Therefore, the Frobenius theorem implies that the horizontal lifts to $\mathbb{S}$:

$$L_0 = X_1 - \lambda X_2 + l_0 \partial_\lambda, \quad L_1 = X_2 - \lambda X_3 + l_1 \partial_\lambda \tag{7.1}$$

satisfy $[L_0, L_1] = \alpha L_0 + \beta L_1$ for some α, β which are linear in λ. The functions l_0 and l_1 are third order in λ, because the Möbius transformations of $\mathbb{C}P^1$ are generated by vector fields quadratic in λ.

To find the third-order ODE corresponding to $([h], D)$ we construct two-independent solutions $x(t_a, \lambda)$, $y(t_a, \lambda)$ to the pair of linear PDEs:

$$L_0 f = L_1 f = 0,$$

and eliminate λ between x and y. This gives $y = Z(x, t^a)$. Now we follow the prescription given in the introduction to produce the third-order ODE. Both invariants (1.7) and (1.8) will be satisfied as a consequence of the EW condition.

As an example, consider the Einstein–Weyl $(++-)$ structure on Thurston's Nil manifold $S^1 \times \mathbb{R}^2$ [17,18] given by

$$h = \alpha^2(\mathrm{d}t^2 + t^1\,\mathrm{d}t^3)^2 - 4\,\mathrm{d}t^1\mathrm{d}t^3, \qquad \omega = \alpha^2(\mathrm{d}t^2 + t^1\mathrm{d}t^3).$$

Choose the Lax pair:

$$L_0 = \partial_1 + \alpha^{-1}\lambda\partial_2, \qquad L_1 = -\alpha^{-1}\partial_2 - \lambda(\partial_3 - t^1\partial_2) + \alpha\lambda\partial_\lambda,$$

so that $[L_0, L_1] = 0$. We find a kernel of (L_0, L_1) to be

$$x = \lambda + \alpha t^3, \qquad y = \lambda t^1 - \alpha t^2 - \alpha^{-1}\ln\lambda,$$

so that the totally geodesic surfaces are given by $y = Z(x, t^a)$ with

$$Z(x, t_a) = (x - \alpha t^3)t^1 - \alpha t^2 - \alpha^{-1}\,\ln(x - \alpha t^3).$$

The resulting third-order ODE is

$$y''' = 2\sqrt{\alpha}(y'')^{3/2}$$

which is a special case of our general example (2.7) with $n = 3$, and (t^1, t^2, t^3) redefined.

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Appendix A

A.1. Determining P and Q

The first step is to calculate recursive formulae for a_{ij}. This yields $a_{00} = 1$, and

$$a_{ik} = 0, \quad \text{for } k > i, \qquad a_{(i+1)k} = (a_{ik})' + kQa_{ik} + (n-k)a_{i(k-1)} + (k+1)a_{i(k+1)}P,$$
$$0 \le k \le i-1, \qquad a_{(i+1)i} = (a_{ii})' + iQa_{ii} + (n-i)a_{i(i-1)},$$
$$a_{(i+1)(i+1)} = (n-i-1)a_{ii}.$$

These relations give

$$a_{ii} = \frac{(n-1)!}{(n-i-1)!}, \qquad a_{(i+1)i} = \frac{(i+1)i}{2}Qa_{ii}, \qquad a_{n(n-2)} = \alpha P + \beta Q^2 + \gamma Q',$$

where

$$\alpha = \frac{n(n+1)!}{6},\ \beta = \frac{n!}{24}(3n-5)(n-1)(n-2),\ \gamma = \frac{n!}{6}(n-1)(n-2).$$

Solving the last two equations in (2.5) corresponding to $j = n-1$, and $j = n$ for P and Q yields (2.6).

A.2. Conditions for F

We shall give explicit forms of (1.10) for $n = 5$:

$$\begin{aligned}
0 = {} & -\frac{1}{5}\frac{d^4}{dx^4}F_4 - \frac{2}{25}F_3\frac{d^2}{dx^2}F_4 - \frac{7}{25}\left(\frac{d}{dx}F_4\right)\frac{d}{dx}F_3 - \frac{28}{3125}(F_4)^5 + \frac{16}{25}\left(\frac{d}{dx}F_4\right)\frac{d^2}{dx^2}F_4 \\
& - F_0 + \frac{8}{25}F_4\frac{d^3}{dx^3}F_4 - \frac{1}{5}F_2F_3 - \frac{7}{100}F_2(F_4)^2 + \frac{1}{5}F_2\frac{d}{dx}F_4 - \frac{11}{125}F_4(F_3)^2 \\
& - \frac{141}{2500}F_3(F_4)^3 + \frac{137}{1250}(F_4)^3\frac{d}{dx}F_4 - \frac{9}{25}F_4\left(\frac{d}{dx}F_4\right)^2 - \frac{9}{50}F_4\frac{d^2}{dx^2}F_3 \\
& - \frac{103}{500}(F_4)^2\frac{d^2}{dx^2}F_4 + \frac{101}{1000}(F_4)^2\frac{d}{dx}F_3 + \frac{7}{50}F_3\frac{d}{dx}F_3 + \frac{1}{5}\frac{d^3}{dx^3}F_3 + \frac{28}{125}F_4F_3\frac{d}{dx}F_4, \\
0 = {} & -12F_2 - \frac{36}{5}F_3F_4 - \frac{48}{25}(F_4)^3 + \frac{72}{5}F_4\frac{d}{dx}F_4 - 12\frac{d^2}{dx^2}F_4 + 18\frac{d}{dx}F_3, \\
0 = {} & \frac{102}{25}F_4\frac{d^2}{dx^2}F_4 + \frac{18}{5}\frac{d^2}{dx^2}F_3 - \frac{16}{5}\frac{d^3}{dx^3}F_4 + \frac{68}{25}\left(\frac{d}{dx}F_4\right)^2 - \frac{4}{625}(F_4)^4 - \frac{16}{25}(F_3)^2 \\
& - \frac{34}{125}F_3(F_4)^2 + \frac{4}{25}F_3\frac{d}{dx}F_4 - \frac{172}{125}(F_4)^2\frac{d}{dx}F_4 - 4F_1 - \frac{2}{5}F_2F_4 - \frac{9}{5}F_4\frac{d}{dx}F_3 = 0.
\end{aligned}$$

The conditions for $n > 5$ can be written down using recursive relations and MAPLE, but the resulting formulae are very long, and inconclusive.

A.3. Calculations leading to a proof of Theorem 1.3

We collect the terms in $g' - \Lambda g = 0$ by order of derivatives (e.g. $\mathrm{d}y\,\mathrm{d}p\,\mathrm{d}r$ has order $4 = 0 + 1 + 3$):

- Sixth order:

$$18F_3 + 2\pi + 2\theta + 2\phi - 18\Lambda = 0, \tag{A.1}$$

$$-18F_3 + \phi + 9\Lambda = 0, \tag{A.2}$$

$$\pi + 6\Lambda = 0, \tag{A.3}$$

$$-8F_3 + \zeta + \theta + 8\Lambda = 0, \tag{A.4}$$

$$3\zeta + \pi - 3\Lambda = 0. \tag{A.5}$$

- Fifth order:

$$-18F_2 + 2\mu + \nu + \phi' + \phi F_3 - \Lambda\phi = 0, \tag{A.6}$$

$$18F_2 + 2\lambda + 2\mu + \pi' - \Lambda\pi = 0, \tag{A.7}$$

$$\lambda + 2\nu + \theta F_3 + \theta' - \Lambda\theta = 0, \tag{A.8}$$

$$-8F_2 + 4\kappa + \lambda + \zeta' - \Lambda\zeta = 0. \tag{A.9}$$

- Fourth order:

$$-18F_1 + \xi + 3\eta + \nu F_3 + \nu' - \Lambda\nu = 0, \tag{A.10}$$

$$\xi + \phi F_2 + \mu' - \Lambda\mu = 0, \tag{A.11}$$

$$18F_1 + 3\epsilon + 2\xi + \theta F_2 + \lambda' - \Lambda\lambda = 0, \tag{A.12}$$

$$-8F_1 + \epsilon + \kappa' - \Lambda\kappa = 0. \tag{A.13}$$

- Third order:

$$-18F_0 + \delta + \eta F_3 + \eta' - \Lambda\eta = 0, \tag{A.14}$$

$$18F_0 + 2\gamma + 3\delta + \nu F_2 + \phi F_1 + \xi' - \Lambda\xi = 0, \tag{A.15}$$

$$-8F_0 + 2\gamma + \theta F_1 + \epsilon' - \Lambda\epsilon = 0. \tag{A.16}$$

- Second order:

$$3\beta + \nu F_1 + \theta F_0 + \gamma' - \Lambda\gamma = 0, \tag{A.17}$$

$$\beta + \eta F_2 + \phi F_0 + \delta' - \Lambda\delta = 0. \tag{A.18}$$

- First order:

$$4\alpha + \eta F_1 + \nu F_0 + \beta' - \Lambda\beta = 0. \tag{A.19}$$

- Zeroth order:

$$\eta F_0 + \alpha' - \Lambda\alpha = 0. \tag{A.20}$$

A.4. Conditions for F leading to a proof of Theorem 1.4

$$B = P_3, \qquad C = Q_3, \quad A = P_2 + QP_3 - 2Q_3P - 2\frac{\mathrm{d}}{\mathrm{d}x}P_3,$$

where P, Q are given by (2.6):

$$2\frac{\mathrm{d}A}{\mathrm{d}x} - Q_0 + (n-1)\frac{\mathrm{d}}{\mathrm{d}x}\left(BQ + \frac{\mathrm{d}}{\mathrm{d}x}B\right) = 0, \tag{A.21}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}C + 2A - Q_1 + \frac{\mathrm{d}}{\mathrm{d}x}(CQ) + (n-1)QB + 2n\frac{\mathrm{d}B}{\mathrm{d}x} = 0, \tag{A.22}$$

$$CQ + 2\frac{\mathrm{d}C}{\mathrm{d}x} - Q_2 + (n+1)B = 0, \tag{A.23}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} - Q\frac{\mathrm{d}A}{\mathrm{d}x} - P_0 + (n-1)\frac{\mathrm{d}}{\mathrm{d}x}(BP) + (n-1)P\frac{\mathrm{d}B}{\mathrm{d}x} = 0, \tag{A.24}$$

$$2\frac{\mathrm{d}A}{\mathrm{d}x} - P_1 - QA + \frac{\mathrm{d}^2B}{\mathrm{d}x^2} - Q\frac{\mathrm{d}B}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}(CP) + P\frac{\mathrm{d}C}{\mathrm{d}x} + 2(n-1)BP = 0. \tag{A.25}$$

References

[1] M.A. Akivis, V.V. Goldberg, Conformal and Grassmann structures, Diff. Geom. Appl. 8 (1998) 177–203.
[2] T.N. Bailey, M.G. Eastwood, Complex Paraconformal manifolds, their differential geometry and twistor theory, Forum Math. 3 (1991) 61–103.
[3] R.L. Bryant, Two exotic holonomies in dimension four, path geometries, and twistor theory, Proc. Symp. Pure Math. 53 (1991) 33–88.
[4] R.L. Bryant, Private communication.
[5] E. Cartan, Sur les varietes a connexion projective, Bull. Soc. Math. France 52 (1922) 205–241.
[6] E. Cartan, La Geometria de las Ecuaciones Diferenciales de Tercer Orden, Rev. Mater. Hispano-Am. 4 (1941) 1–31.
[7] E. Cartan, Sur une classe d'espaces de Weyl, Ann. Sci. Ecole Norm. Suppl. 60 (1943) 1–16.
[8] S.S. Chern, The Geometry of the Differential Equation $y''' = F(x, y, y', y'')$. In Selected Papers, Springer-Verlag, Berlin, 1978.
[9] B. Doubrov, Contact trivialization of ordinary differential equations, Diff. Geom. Appl. (2001) 73–84.
[10] V. Dryuma, The Riemann extensions in theory of differential equations and their applications, Mater. Fiz. Anal. Geom. 10 (2003) 307–325.
[11] M. Dunajski, L.J. Mason, K.P. Tod, Einstein–Weyl geometry, the dKP equation and twistor theory, J. Geom. Phys. 37 (2001) 63–92.
[12] M. Dunajski, L.J. Mason, S. Gindikin, Solitons and admissible families of rational curves in twistor spaces, Nonlinearity 18 (2005) 543–557.
[13] S. Frittelli, C. Kozameh, T. Newman, Differential geometry from differential equations, Commun. Math. Phys. 223 (2001) 383–408.
[14] N. Hitchin, Complex manifolds and Einstein's equations, in: H.D. Doebner, T.D. Palev (Eds.), Twistor Geometry and Non-linear Systems, LNM 970, Springer, 1982.
[15] K. Kodaira, On stability of compact submanifolds of complex manifolds, Am. J. Math. 85 (1963) 79–94.
[16] B. McKay, Rational curves and ordinary differential equations, 2005, math.DG/0507087.
[17] H. Pedersen, K.P. Tod, Three-dimensional Einstein–Weyl geometry, Adv. Math. 97 (1993) 74–109.
[18] K.P. Tod, Einstein–Weyl spaces and third order differential equations, J. Math. Phys. 41 (2000) 5572.
[19] E.J. Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, B.G. Teubner, Leipzig, 1906.
[20] K.W. Wünschmann, Inaug. Dissert, Teubner, Leipzig, 1905.